and the EA terms in the other coefficients are dropped. This shows that the usual assumption of inextensional buckling is correct if there are no tangential loads or supports.

Greater simplification may be made if the coefficient a in Eq. (21) is zero, for then the quadratic degenerates and the modes of buckling become uncoupled. This occurs if $x_c = y_c = x_b = I_x/Ar_c = I_p/Ar_c = 0$. Since $I_p = 0$ requires that both I_x and I_x be zero, the solution would not be very significant from a design standpoint. However, if $Ar_c \gg I_p$, approximate solutions may be obtained by neglecting the terms I_x/Ar_c and I_x/Ar_c. Therefore, with $a = 0$,

$$q = \frac{1}{C_2}\left(-C_1 + \frac{C_3^2}{C_7} + \frac{C_4^2}{C_{12}}\right) \tag{25}$$

The determination of the effective elastic support for the ring is the subject of future research. However, in general the spring support may be obtained by choosing a spring constant that produces the same internal strain energy as the shell does upon distortion.

A formulation of an effective elastic spring constant is made by Czerwenka[2] for a shell beyond initial buckling. The value is in general a function of the buckling mode m.

Conclusion

The buckling pressure for a shell-supported ring has been derived and is represented by Eq. (21). The equation may be simplified in the case where the shear center S coincides with the centroid C, the loading is through the centroid, and $Ar_c \gg I_p$. The result of such simplification is given in Eq. (26). Table 1 gives explicit equations derived from the latter case.

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Technical Comments

Comment on "Low-Altitude, High-Speed Handling and Riding Qualities"

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The following offers an explanation for the "PIO limits" reported by A'Harrah[1] and uncovers a common deficiency in ground and flight simulation studies of handling qualities, one which can easily be avoided when appreciated.

For a sustained Pilot-Induced Oscillation (PIO) it is postulated that, because of the regular nature of the oscillation seen or felt by the pilot, he will synchronize his output and eliminate his usual reaction-time delay.[2] Under such circumstances the pilot's sinusoidal describing function is a simple gain. Assuming that his primary response is to visual pitch-attitude cues, the pertinent longitudinal pilot-vehicle open-loop describing function (neglecting phugoid motions) at any limit cycling frequency ω, where $s = j\omega$, is [3]

$$Y_p(s)\frac{\theta}{\delta_e}(s) = \frac{K_pM_\delta[s + (1/T_{\theta_2})]}{s(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2)} \tag{1}$$

Accordingly, the closed loop characteristic equation is given by

$$\Delta' = s(s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2) + K_pM_\delta[s + (1/T_{\theta_2})]$$
$$= s^3 + s^2(2\zeta_{sp}\omega_{sp}) + s(\omega_{sp}^2 + K_pM_\delta) + K_pM_\delta(1/T_{\theta_2}) \tag{2}$$

where the usual factored form of Δ' is expressed as

$$\Delta' = [s + (1/T_c)](s^2 + 2\zeta_{sp}'\omega_{sp}'s + \omega_{sp}'^2)$$
$$= s^3 + s^2[(1/T_c) + 2\zeta_{sp}'\omega_{sp}'] + s[\omega_{sp}'^2 + (2\zeta_{sp}'\omega_{sp}'/T_c)] + \omega_{sp}'^2/T_c \tag{3}$$

Since a sustained oscillation implies $\zeta' = 0$, the conditions for PIO [and the only condition for which Eq. (3) has meaning] correspond to those for which $\zeta_{sp}'\omega_{sp}'$ is zero. That is equating the s^2 coefficients of Eqs. (2) and (3),

$$2\zeta_{sp}'\omega_{sp}' + (1/T_c) = 2\zeta_{sp}\omega_{sp}$$
$$2\zeta_{sp}'\omega_{sp}' = 2\zeta_{sp}\omega_{sp} - (1/T_c) = 0 \tag{4}$$

Obviously if $1/T_c$ can become equal to $2\zeta_{sp}\omega_{sp}$, the system *can* be driven unstable by sufficiently high gain. To get a better feeling for such possibilities, consider the root locus plot of Eq. (1) given in Fig. 1. The relationship of Eq. (4) is graphically illustrated here, and conclusions as to the maximum value of $1/T_c$ and minimum value of $2\zeta_{sp}'\omega_{sp}'$ are clearly

$$(1/T_c)_{\max} \to (1/T_{\theta_2})$$
$$(2\zeta_{sp}'\omega_{sp}')_{\min} \to 2\zeta_{sp}\omega_{sp} - (1/T_{\theta_2}) \tag{5}$$

Probable values for the right side of Eq. (5) can be obtained by considering the approximate factors of Ref. 4,

$$2\zeta_{sp}\omega_{sp} \doteq -(Z_w + M_q + M_{\dot\alpha})$$
$$(1/T_{\theta_2}) \doteq -Z_w + (Z_\delta/M_\delta)M_w \tag{6}$$

so that

$$2\zeta_{sp}\omega_{sp} - (1/T_{\theta_2}) \doteq -M_q - M_{\dot\alpha} - (Z_\delta/M_\delta)M_w \tag{7}$$
$$\doteq \frac{\rho SU_0c^2}{4I_y}\left[-C_{M_q} - C_{M_{\dot\alpha}} + \frac{2k_y^2}{cl_\delta}C_{M_\alpha}\right]$$

where $l_\delta = cC_{M_\delta}/C_{L_\delta}$ is the effective elevator control arm measured positive *forward*. For conventional tail-aft airplanes with some static margin (l_δ and C_{M_α} both negative) the bracketed terms of Eq. (7) are always positive; for canard control the contribution of the C_{M_α} term will be negative because of the positive l_δ. In the latter case it is conceivable that the entire right side of Eq. (7) could be negative. However, a general observation is that it will take a very unusual configuration with a small l_δ and low values of $-C_{M_q}$ (which, for conventional airplanes, is usually an order of magnitude greater than C_{M_α}), etc., to make the value of $2\zeta_{sp}\omega_{sp} - 1/T_{\theta_2}$ negative, as sketched in Fig. 1. Therefore, *only* for such unusual configurations is there a possibility of driving $\zeta'\omega'$ to zero to achieve a sustained PIO.

The foregoing demonstrates that for real airplanes with negligible control system dynamics (including nonlinear ele-

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ments) longitudinal PIO's involving attitude control only are essentially impossible.† However, in variable-stability flight testing and in ground-based simulation studies where the general practice is to hold Z_w constant and vary ζ_{sp}, ω_{sp}, stick force and displacement per g, etc., artificial relationships between $\zeta_{sp}\omega_{sp}$ and $1/T_{\theta_2}$ can lead to PIO's of the simple type under consideration here. For example, Fig. 2 of A'Harrah's paper,[1] reproduced here (also as Fig. 2), shows the cross-hatched PIO "limits" for the lower left region of the ζ,ω plane. The data were obtained for a fixed value of Z_w corresponding to $1/T_{\theta_2} = 3.22\ \text{sec}^{-1}$ for all the conditions tested. The theoretical boundary for zero ζ_{sp}' as given by Eq. (5) is superimposed on the original plot. It may be seen that there is general agreement between the predicted possible "simple" PIO region to the left of the boundary and the observed region. The fact that the experimental region for very light stick-force gradients lies somewhat to the right of the theoretical boundary is evidence of additional dynamics—in this case probably nonlinear effects due to the high breakout force to stick gradient, 1.2 lb/1.0 lb/g.

Although the PIO region of Fig. 2 is thus shown to be dependent on artificial relationships between $1/T_{\theta_2}$ and $\zeta_{sp}\omega_{sp}$, there *are* flight test examples of PIO's in precisely the same region (e.g., Refs. 5 and 6). The degree to which all such situations are dependent on control system contributions, linear or nonlinear, is not exactly known. Nevertheless it appears to be true that longitudinal PIO's were nonexistent (or not reported) until the advent of modern hydraulically powered elevator actuation systems. The PIO reported in Ref. 6 can in fact be traced directly to the *linear* (lag) contribution of the hydraulic system, which was measured and reported. It appears therefore that, except for very unusual configurations, longitudinal PIO's can be sustained only for conditions in which control system dynamics are a contributing cause.

As a final observation it must be emphasized that the foregoing is but one example of improper simulation due to inadequate consideration of numerator dynamics; there are many other examples in the pertinent literature (and the writer's own organization is to blame for at least one of these). The specific importance of the numerator dynamics associated with aileron control of bank angle has been theoretically and experimentally established[3] and is not likely to be forgotten. But the short shrift usually given to the proper simulation of numerators in general continues to give anomalous, misleading, and generally inapplicable handling quality results. It is the writer's hope that the fairly dramatic example exposed here will stimulate more widespread interest in simulating the complete numerator-denominator vehicle characteristics.

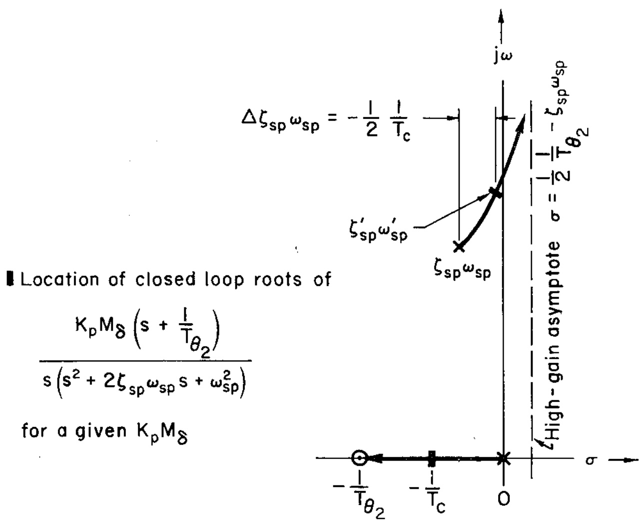

Fig. 1 Root locus plot of Eq. (1).

† It should be emphasized here that, despite accompanying vertical accelerations, attitude cues will be those the pilot primarily uses in attempting to get out of a PIO situation, so the foregoing considerations are valid. Acceleration inputs introduced by the pilot's arm bobweight effect acting through control system friction could be a nonlinear destabilizing influence.

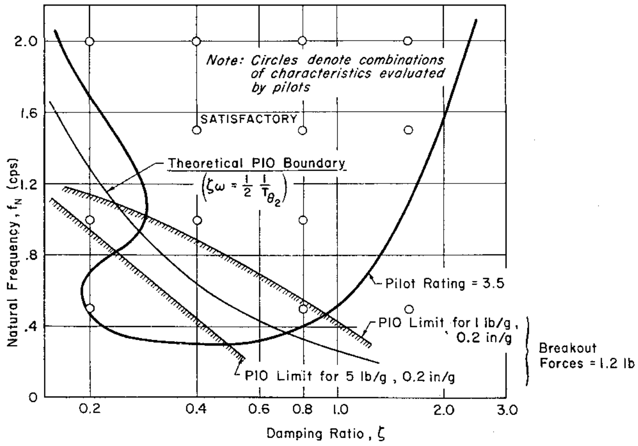

Fig. 2 Comparison of theoretical and observed PIO limits.

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Reply by Author to I. L. Ashkenas

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THE following is a rebuttal to the critical review by Irving L. Ashkenas of the Pilot-Induced Oscillation (PIO) boundaries defined from the dynamic flight simulator research results reported in Ref. 1. The word rebuttal implies an argument which is indeed the intent, though let it be first understood that this author agrees with Ashkenas on the two points developed at length in his critique, namely 1) that for reasonably configured airplanes with negligible control system dynamics and negligible nonlinearities, PIO's involving attitude control only are impossible for pilots who perfectly synchronize their corrective control with the aircraft's attitude change; and 2) that the value of $1/\tau_{\theta_2}$ used in the simulation program is not numerically consistent with the level of short period damping for two of the evaluation points ($\zeta = 0.2$ at $f_n = 0.5$ and 1.0 cps). However, to state that on the basis

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